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Nonuniform Hyperbolicity: Dynamics of Systems with Nonzero Lyapunov Exponents

To Clàudia

and

To Natasha, Irishka and Lenochka

for their patience, encouragement and inspiration

Contents

Preface

Writing this book was a long-term project that has taken several years, and at the early stages Anatole Katok's participation was crucial. He provided us with the text of his unpublished notes with Leonardo Mendoza that served as the basis for the first draft of Chapters 1–5 and parts of Chapters 6–7 of the book. He also fully participated in designing the content of the book in its present form.

In this book we present a self-contained and sufficiently complete description of the modern nonuniform hyperbolicity theory, that is, the theory of dynamical systems whose Lyapunov exponents are not zero. The reader will find all the core results of the theory as well as a good account of its recent developments.

The nonuniform hyperbolicity theory is rich in wonderful ideas and sophisticated techniques, which are widely used in many areas of dynamical systems as well as other areas of mathematics and beyond. The nonuniform hyperbolicity theory is very popular and finds a lot of applications outside mathematics – in physics, biology, engineering, and so on.

Despite (or should we say because of) a tremendous amount of research on the subject, there have been relatively few attempts to summarize and unify the results of the theory in a single manuscript or a survey (see the books [110, 139, 179], the surveys [18, 137, 175], and the lectures [19, 28]). This book is meant to cover this gap. It can be used as a reference book for the theory or as a supporting material for an advanced course on dynamical systems. During the long course of working on this book, we first produced its baby version [20] where we described the core results of the theory and some principal examples, and then we wrote the survey [21] where we presented the contemporary status of the theory.

Since the beginning of the 1970s, the nonuniform hyperbolicity theory has emerged as an independent discipline lying in the heart of the modern theory of dynamical systems. It studies both conservative (volume preserving) and dissipative systems, deterministic as well as random dynamical systems, discrete and continuous-time systems, in addition to cocycles and group actions. The results of this theory have found their way in geometry (e.g., in the study of geodesic flows and Teichmüller flows), in rigidity theory, in the study of some partial differential equations (e.g., the Schrödinger equation and some reaction-diffusion equations), and in the theory of "chaotic" billiards.

Writing a book of such a scope can be deemed as a daunting task and we therefore had to select topics so that personal taste, clearly biased toward our own interests, entered in our choices. As a result, some interesting topics are barely mentioned or not covered at all. In particular, we do not consider random dynamical systems referring the reader to the books [9, 112, 129] and the survey [113] nor dynamical systems with singularities (see the book [110]), in particular, leaving aside the rich theory of chaotic billiards. We restrict ourselves to the case of invertible dynamical systems and thus the theory of nonuniformly expanding maps is not discussed here (see the survey [132]) nor do we include one-dimensional chaotic maps (e.g., the logistic family, see [94]). We touch upon some recent results on Hénon-like attractors related to the study of Sinai–Ruelle–Bowen measures but we do not go deep into the theory of these attractors (see the survey [133]). We mention some results on hyperbolic group actions and refer the reader to [71] for a more complete account.

All the principal results of the nonuniform hyperbolicity theory are presented in the book with complete proofs, although some other results are included without proofs for the sake of completeness.

Most chapters of the book end with notes where the reader can find some remarks of historical and bibliographical nature, comments on some related results, and references for further reading. In no way these notes are meant to present a significant account of the history of the subject or a sufficiently complete list of references.

Acknowledgments

While working on the book, several people helped us in various ways and it is our great pleasure to acknowledge their contributions.

As we mentioned above, it is impossible to overestimate the contribution of Anatole Katok whom we heartily thank for his constant support and guidance.

We also thank Marlies Gerber for providing us with the text of Section 6.4.2 and Theorem 9.2.8 and for some useful remarks.

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We are grateful to Claudia Valls who carefully read several chapters of the book and made many remarks that helped us clarify the presentation.

When the first draft of the book was ready, we sent it to several experts in the field asking for their opinions and comments. We are grateful to Dima Dolgopyat, François Ledrappier, Mark Pollicott, and Federico Rodríguez Hertz for innumerable constructive suggestions that helped us improve the content and presentation of some results of the book as well as extend and enhance the bibliography.

Our special thanks go to Boris Hasselblatt who thoroughly examined the draft

and pointed out many places in the book that needed additional work. Reflecting on his comments we introduced many changes improving the style and exposition of the material; we also added more informal discussions, hopefully making the book more reader friendly.

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Introduction

The goal of this book is to present smooth ergodic theory from a contemporary point of view. Among other things this theory provides a rigorous mathematical foundation for the phenomenon known as *deterministic chaos* – a term coined by Yorke – the appearance of highly irregular, unpredictable,“chaotic” motions in pure deterministic dynamical systems. The main idea beyond this phenomenon is that one can deduce a sufficiently complete description of topological and ergodic properties of the system from relatively weak requirements on its local behavior, known as *nonuniform hyperbolicity conditions*: the reason this theory is also called nonuniform hyperbolicity theory.

It originated in the seminal works of Lyapunov [134] and Perron [164] on stability of solutions of ordinary differential equations. To determine whether a given solution is stable one proceeds as follows. First, the equation is linearized along the solution and then the stability of the zero solution of the corresponding nonautonomous linear differential equation is examined. There are several methods (due to Hadamard [79], Perron [165], Fenichel [70], and Irwin [92]) aimed at exhibiting stability of solutions via certain information on the linear system. The approach by Lyapunov uses a special real-valued function on the space of solutions of the linear system known as the *Lyapunov exponent*. It measures in the logarithmic scale the rate of convergence of solutions so that the zero solution is asymptotically exponentially stable along any subspace where the Lyapunov exponent is negative.

The Lyapunov exponent is arguable the best way to characterize stability: the requirement that the Lyapunov exponent is negative is the weakest one that still guarantees that solutions of the linear system eventually decay exponentially to zero. The price to pay is that stability of the zero solution in this weak sense does not necessarily imply stability of the original solution of the nonlinear equation. The latter can be ensured under an additional and quite subtle requirement known as the *Lyapunov–Perron regularity*.

Verifying this requirement for a given solution may be a very difficult if not virtually impossible task, making verification more a principle than practical matter. This could deem the whole approach useless if not for an important particular case when the differential equation is given by a vector field on a smooth compact Riemannian manifold. In this case, the celebrated Multiplicative Ergodic Theorem, also known as Oseledets theorem, claims that a "typical" solution of the equation is Lyapunov–Perron regular, thus making the difficult task of checking the regularity requirement unnecessary. Here "typical" means that the statement holds for almost every trajectory with respect to a finite Borel measure invariant under the flow generated by the vector field.

A principal application of Oseledets theorem in the context of smooth dynamical systems is that the Lyapunov exponent alone can be used to characterize stability of trajectories. Building upon this idea, in the beginning of 1970s Pesin introduced the class of systems whose Lyapunov exponent is nonzero along almost every trajectory with respect to some *smooth invariant measure* (i.e., a measure, which is equivalent to the Riemannian volume) and then he developed the stability theory (constructing local and global stable and unstable manifolds; see Section 7.5), as well as described their ergodic properties (including ergodicity, K- and Bernoulli properties; see Chapter 9). The collection of these results is known as Pesin's theory (see [18]). A crucial manifestation of this theory is the *formula for the entropy* connecting the measure-theoretic entropy of the system with its Lyapunov exponent (see Chapter 10). It should be pointed out that these results require that the system is of class of smoothness $C^{1+\alpha}$ for some $\alpha > 0$ and that they may indeed fail if the system is only of class C^1 (see Section 7.8).

Unlike classical uniformly hyperbolic systems (i.e., Anosov or more general axiom A systems) where contractions and expansions are *uniform everywhere* on a compact invariant set, Pesin's theory deals with systems satisfying the substantially weaker requirement that contractions and expansions occur *asymptotically almost everywhere* with respect to a smooth invariant measure. Because this requirement is weak, there are no topological obstructions for the existence of such systems on any phase space. Indeed, any smooth compact Riemannian manifold (of dimension ≥ 2 in the discrete-time case and of dimension ≥ 3 in the continuous-time case) admits a volume preserving system whose Lyapunov exponent is nonzero almost everywhere (see Sections 11.4 and 11.5). It is therefore remarkable that such a weak requirement ensures highly nontrivial ergodic and topological properties of the system.

A small perturbation of a diffeomorphism with nonzero Lyapunov exponents (in the C^r topology, $r > 1$) may not bear the same properties – the price to pay for the great generality of the nonuniformly hyperbolic theory. However, experts believe that nonuniformly hyperbolic *conservative* systems (i.e., systems preserving a smooth measure, in particular, volume preserving) are *typical* in some sense. This is reflected in the following conjectures: (We consider the case of systems with discrete time.)

1. Let f be a C^r, $r > 1$, volume preserving diffeomorphism of a smooth compact Riemannian manifold M. Assume that the Lyapunov exponent of f is nonzero along almost every trajectory of f. Then there exists a neighborhood $\mathcal{U}$ of f in the space of C^r volume preserving diffeomorphisms of M and a residual subset $\mathcal{A} \subset \mathcal{U}$ such that for every $g \in \mathcal{A}$ the Lyapunov exponent of g is nonzero along every orbit in a subset of positive volume.

2. Let f be a C^r, $r > 1$, volume preserving diffeomorphism of a smooth compact Riemannian manifold M. Then arbitrarily close to f in the space of C^r volume preserving diffeomorphisms of M, there exists a diffeomorphism g whose Lyapunov exponent is nonzero along every orbit in a subset of positive volume.

We stress that the assumption $r > 1$ is crucial as the conjectures fail if $r = 1$ due to a recent result of Bochi and Viana [28]. So far there has been little progress in solving these conjectures (see Section 11.7). On the positive side, crucial results on genericity of hyperbolic cocycles over dynamical systems have been recently obtained by Viana [221].

A persistent obstruction to nonuniform hyperbolicity is presence of elliptic behavior (see [232, 233]). For example, for area preserving surface diffeomorphisms, as predicted by KAM theory, elliptic islands survive under small perturbations of the system. Numerical studies of such maps suggest that in this case elliptic islands coexist with what appears to be a "chaotic sea" – an ergodic component of positive area with nonzero Lyapunov exponents (see [135, 136]). In fact, one often considers a one-parameter family of area preserving surface diffeomorphisms, which starts from a completely integrable (nonchaotic) system and evolves eventually into a completely hyperbolic (chaotic) one demonstrating, for intermediate values of the parameter, the appearance of elliptic islands gradually giving way to a "chaotic sea". For billiard dynamical systems, coexistence of elliptic and hyperbolic behavior has been shown for the so-called "mushroom billiards" (see [41]). In the category of smooth maps, establishing coexistence is arguably one of the most difficult problems in the theory of dynamical systems. A simple but somewhat "artificial" example of coexistence was constructed in [183] (see also [130] and Section 6.6; for a more elaborate construction see [90]). Much more complicated examples where coexistence is expected are (1) the famous standard map (also known as the Chirikov–Taylor map; see [51] and [188, Section 8.5]) and (2) automorphisms of real $K3$ surfaces (see [151]).

The requirement that the Lyapunov exponent is nonzero along almost every trajectory with respect to an invariant Borel probability measure – such a measure is said to be *hyperbolic* – is equivalent to the fact that the system is nonuniformly hyperbolic. Thus nonuniform hyperbolicity can be viewed as presence of hyperbolic invariant measures leading to challenging problems of studying ergodic and topological properties of general (not necessarily smooth) hyperbolic measures as well as of constructing some *natural* hyperbolic measures.

A general hyperbolic measure does not have "good" ergodic properties. (Simply note that *any* invariant measure on a horseshoe is hyperbolic.) It is therefore quite remarkable that hyperbolic measures have abundance of topological properties whose study was initiated in the work of Katok [101] (see Chapters 14 and 15). For example, the set of hyperbolic periodic orbits is dense in the support of the measure. Surprisingly, general hyperbolic measures asymptotically have local product structure (similar to the one of Gibbs measures on horseshoes) and one can compute their Hausdorff dimension and entropy. The formula for the entropy of a general hyperbolic measure due to Ledrappier and L.-S. Young is a substantial generalization of the entropy formula for smooth hyperbolic measures but unlike the latter, it involves quite subtle characteristics of the measure other than the Lyapunov exponent.

Smooth measures form an important yet particular case of natural hyperbolic measures. The latter were introduced by Ledrappier as an extension to nonuniformly hyperbolic systems of the Sinai–Ruelle–Bowen (SRB) measures for classical uniformly hyperbolic attractors (see Chapter 13). These *generalized* SRB measures describe the limit distribution of the time averages of continuous functions along forward orbits for a set of initial points of positive Lebesgue measure in a small neighborhood of the attractor. According to a result by Ledrappier and Strelcyn, these measures can be characterized as being the only measures for which the entropy formula of Pesin holds. Ledrappier showed that the methods used in studying ergodic properties of smooth hyperbolic measures can be adjusted to describe ergodic properties of SRB measures.

Constructing SRB measures for nonuniformly hyperbolic systems is a difficult problem. Beyond uniform hyperbolicity, there are very few examples, of which best known are Hénon-like attractors, where existence of SRB measures was rigorously shown. L.-S. Young has introduced a class of dynamical systems with nonzero Lyapunov exponents, which admit the so-called Young's tower. For these systems, she established existence of SRB measures (see Section 13.3).

The recent theory of Hénon-like diffeomorphisms (see [25, 26, 219, 222, 223]) suggests the following approach to the genericity problem for nonuniformly hyperbolic *dissipative* systems: given a one-parameter family of C^2 diffeomorphisms f_a, $a \in [\alpha, \beta]$ with a trapping region R (i.e., R is an open set for which $\overline{f_a(R)} \subset R$ for any $a \in [\alpha, \beta]$), there exists a set $A \subset [\alpha, \beta]$ of positive Lebesgue measure such that for every $a \in A$, the diffeomorphism f_a possesses an SRB measure supported on the attractor $\Lambda_a = \bigcap_{n>0} f_a^n(R)$.

Evaluating Lyapunov exponents by a computer is a relatively easy procedure and in many models in science, the absence of zero exponents can be shown numerically. This is often viewed as a convincing evidence that the system under investigation exhibits chaotic behavior. In mathematics, several "artificial" examples of systems with nonzero exponents have been constructed (and the reader can find most of them in Chapter 6) and for some interesting "natural" dynamical systems (e.g., geodesic flows on nonpositively curved manifolds and Teichmüller geodesic flows; see Chapter 12) absence of zero exponents have been shown. In addition, various powerful methods have been developed (e.g., cone and Lyapunov function techniques; see Chapter 4) that allow one to verify whether a given dynamical system has some positive Lyapunov exponents.

Many results of the nonuniform hyperbolicity theory hold in greater generality than for actions of single dynamical systems and wherever possible we describe the theory with this view in mind. For example, the linear hyperbolicity theory (including the theory of Lyapunov exponents and its principal result – the Multiplicative Ergodic Theorem) is presented for linear cocycles over dynamical systems (or even over higher-rank Abelian actions), and the stable manifold theory is developed for sequences of diffeomorphisms. Even in the case of an action of a single dynamical system, we consider a more general case of nonuniform *partial* hyperbolicity where the requirement that the values of the Lyapunov exponent are *all* nonzero is replaced by a weaker one that *some* of the values of the Lyapunov exponent are nonzero. Such generalizations require some more complicated techniques and tools from various areas of mathematics to be used and thus make the exposition more complicated but they substantially broaden applications and show the great power of the nonuniform hyperbolicity theory.

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