The Oblique Derivative Problem.
The Poincaré Problem.

Boris P. Paneah

Preface

This book is a monograph i.e. it is devoted to the study of a single theme: the solvability of the oblique derivative problem when the vector field of the problem is tangent to the boundary of a domain on some subset. This is a typical degenerate elliptic boundary problem. The book covers all essential known results related to this problem, although some results are only formulated. The interest to the oblique derivative problem is conditioned by its outward simplicity and geometric intuitiveness. These features allow to trace easily the connection between the deviation of the formal properties of the problem from the ellipticity and the failure of the qualitative picture of the solvability of the problem in comparison with the elliptic situation.

On the other hand due to Poincaré is well known that the oblique derivative problem arises naturally when determining the gravitational field of the Moon, the Earth and the other celestial bodies. It is symptomatic that the vector fields appearing are always tangent to these bodies' surfaces at the points of some curves. In the last 30 years, numerous investigations of the gravitational field of the Moon and the Earth were carried out using both the computational methods (the simple layer model) and the satellite data. Time will show whether the results stated in this book will also have applications to this field.

In preparing this book I am indebted to many people. First of all I would like to express my hearty thanks to those my friends and colleagues who read various parts of the text and gave me suggestions, advices and support. In particular, I wish to thank Professors Robert Brooks, Michael Cwikel, Nadav Liron, Yehuda Pinchover, Allan Pinkus, Simeon Reich, Aizik Volpert and Bronislaw Wajnryb from the Technion, Professor Victor Ivrii from the University of Toronto and Professor Roger Cooke from the University of Vermont for their help. I am deeply indebted to Professor Jurii Lyubich from the Technion, Professor Peter Popivanov from the Institute of Mathematics of Sofia and Professor Leonid Volevich from the Keldysh Institute of Moscow for their detailed and constructive criticism. They not only found a number of mistakes but suggested many improvements to the text.

The major part of this book was prepared at the Technion – Israel Institute of Technology, where I began to work in 1990. The atmosphere of calm and benevolence reigning here together with various forms of support make for a productive creative work in a great extent. A beneficial influence of this atmosphere I have felt in the high degree. I take this opportunity to thank the Technion for all support I have received.

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I owe a great debt of gratitude to my wife Ida, not only for her understanding, patience and moral support, but also for invaluable help during the preparation of this book. In our Age of Enlightenment, when book-printing is simplified to the extent that an author has to compose his book by himself, she took upon herself all the technical part of preparing the book, although she had to learn TEX, LaTEX and all other sorts of TEXs for this purpose.

Introduction

The theory of solvability of general elliptic boundary problems in a bounded domain $\Omega \subset \mathbb{R}^{n+1}$ was essentially complete by the mid-1960s. The central result of this theory (in the context of $H^{(s)}$ spaces) is the theorem asserting that this is a Fredholm problem whose solutions are regular and that the (always finite) index of the problem is independent of the number s. At present, however, much less is known regarding degenerate elliptic boundary problems of general form, that is, boundary problems for elliptic differential equations in the closure $\overline{\Omega}$ of a domain Ω. For these problems the ellipticity conditions (the Schapiro-Lopatinskii condition or the coercivity condition) fail to hold on certain subsets of the boundary $M = \partial\Omega$. The classical example of such a problem is the oblique derivative problem for a second-order elliptic differential operator $\mathcal{L}$

$$\mathcal{L}u = F \text{ in } \Omega, \quad \partial u / \partial l = f \text{ on } M, \qquad (1)$$

when the vector field l is tangent to the boundary M on some subset µ. In this case the solvability properties of the problem (1) change abruptly in comparison with the situation when the vector field l is transversal to the manifold M at every point (so that the problem (1) becomes elliptic). The nature of these changes is very unstable and involves the size and structure of the set μ (though it does involve them) less than the behavior of the vector field l in a neighborhood of the set μ.

We first assume that the tangential set μ is a closed submanifold in M of codimension 1 and l is transversal to μ. In that case μ is the union of connected components $\mu_1, \mu_2, \mu_3, \mu_4$ such that the field l reverses orientation relative to M on each of the submanifolds μ_1 and μ_2, but not on μ_3 and μ_4. These were the assumptions under which the degenerate oblique derivative problem was studied in the 1960s and the early 1970s, in the original papers devoted to solving it. It turned out that if at least one of the submanifolds μ_1 and μ_2 is nonempty, the index of problem (1) is infinite (a case that had been unheard of up to that time in the theory of elliptic equations). At the same time, if $\mu = \mu_3$ and $\mu = \mu_4$, the problem remains a Fredholm problem (as in the elliptic case). Thus in general to obtain a well-posed statement the oblique derivative problem must be modified. It is, of course, desirable to do this in such a way that the original conditions (1) are violated as little as possible. In the papers mentioned above (which, as it happened, used very different methods) such a modification was proposed, to the effect that the value of the required solution was prescribed on each emergent submanifold μ_1, whereas on each submergent submanifold μ_2 the solution has a discontinuity (with a finite jump on μ_2).

In this case the following results were obtained: The zero index theorem for the modified problem; a description of the kernel and co-kernel of the problem; conditions for unique solvability; the strong maximum principle for discontinuous solutions; conditions under which the solution of the modified problem is simultaneously a solution of the original problem (1); precise results on the smoothness of the solutions; necessary and sufficient conditions for coercive estimates.

These results were subsequently generalized to the case of an arbitrary closed set μ, including a subset of full dimension n subject only to the condition that μ does not contain any complete semitrajectory of the field l. It was established that if the field l does not reverse orientation relative to the boundary M, then in relation to solvability the oblique derivative problem preserves all the principal features of elliptic problems. The nonelliptic nature of this problem leads only to a worsening of the regularity of its solutions. If the field l does reverse orientation relative to M, remaining transversal to some smooth submanifold μ_{12}, then, as above, by allowing a discontinuity of the solution on one component μ_2 and prescribing the value of the solution on the other component μ_1, we obtain a Fredholm problem. In contrast to the case when μ is a submanifold of codimension 1, in this situation the submanifolds μ_1 and μ_2 are not uniquely determined, but can be chosen with some degree of arbitrariness. Apart from the Fredholm property all the results listed above were obtained in the case of an arbitrary set μ, including the maximum principle and the description of the functions F and f for which the problem (1) has a classical solution (excluding coercivity and additional smoothness, however).

The present book has a dual purpose: to give as comprehensive an exposition of the results relating to the oblique derivative problem as possible using a unified method, and to give a description of certain contemporary approaches to the study of boundary problems for elliptic equations. Yet another (albeit ulterior) motive for the book is to provide a springboard for a new leap forward in the study of the oblique derivative problem by gathering the majority of the known facts in a single source for the first time. The degenerated oblique derivative problem was brought into existence by Poincaré nearly a century ago in connection with the study of completely realistic physical phenomena (high and low tides on the surface of the earth), but was not solved by him. As a result of the simplicity of its statement and geometric intuitiveness, it became nowadays a sort of testing ground for the latest methods of the theory pseudo-differential operators, Fourier integral operators, and the like). One may assume that the subsequent development of this theory will involve complicating the topological picture of the way in which the ellipticity of the problem (1) degenerates. Who knows what discoveries in analysis await us on this route?!

On the basis of what has been said above, keeping in mind the environment in which one should seek new recruits, the author has striven to make the book accessible to a wide circle of readers, including upperclass undergraduate mathematics majors. For that reason the book opens with Chapter 1, which contains extensive material from different areas of geometry, analysis, functional analysis, and differential equations, of which extensive use will be made both in the description of the results and (especially) in the proofs.

The material for this chapter was chosen so as to minimize the need to refer to additional sources of information and thereby make the book as self-contained as possible. On the other hand, even a short list of the topics discussed in this chapter will enable the reader to understand clearly the minimum amount of information from related areas of mathematics needed nowadays for successful work in the theory of boundary problems.

Yet another step toward accessibility is that, although all the most general results on the solvability of the oblique derivative problem (with an arbitrary set μ) are stated in Chapter 2, their proofs are postponed to Chapter 5. At the same time, all the results given in Chapter 2 are proved in great detail in Chapter 3, only in the technically simplest case when the set of tangency μ is a submanifold. In this situation not only are the proofs much simpler from the point of view of the technique applied, the underlying ideas become absolutely transparent. Indeed, in the author's plan Chapter 3 could serve as the foundation for a one-semester course for undergraduates specializing in partial differential equations. With some independent work on Chapter 1, the students who attend the course will know everything that was known about the oblique derivative problem at the time when μ was still a submanifold. By this stage the students will be able to proceed to independent investigations (see, for example, Problems 1 - 2 in the Appendix).

Chapter 4 stands rather apart from the rest of the book. It is devoted to some rather subtle results involving the possibility of improving information on the smoothness of the solutions of the oblique derivative problem in comparison with the information provided by the general theorems of Chapters 2 and 3. From a technical point of view this material is perhaps the most difficult in the book.

The book closes with Chapter 6, which contains extensive information on results closely related in some sense to those in the preceding chapters, sometimes representing a further development of them. In some cases these results are only stated, but rather detailed proofs are provided for some. However, the facts presented here are always intended to attract the reader's attention to a new problem.

A separate list of problems, very far from complete, is given in the Appendix. We add also that a brief survey of the history of the oblique derivative problem can be obtained by reading the bibliographical notes.

@article{Kozlov2014ObliqueDP, title={Oblique derivative problem for non-divergence parabolic equations with discontinuous in time coefficients}, author={Vladimir Kozlov and Alexander D. Nazarov}, journal={arXiv: Analysis of PDEs}, year={2014}, pages={177-191} }. Vladimir Kozlov, Alexander D. Nazarov. Published 2014. Mathematics. arXiv: Analysis of PDEs. We consider an oblique derivative problem for non-divergence parabolic equations with discontinuous in t coefficients in a half-space. We obtain weighted coercive estimates of solutions in anisotro View PDF on ArXiv. We consider an oblique derivative problem for non-divergence parabolic equations with discontinuous in t coefficients in a half-space. We obtain weighted coercive estimates of solutions in anisotropic Sobolev spaces. We also give an application of this result to linear parabolic equations in a bounded domain. In particular, if the boundary is of class ${\cal C}^{1,\delta}$, $\delta\in (0,1]$, then we present a coercive estimate of solutions in weighted anisotropic Sobolev spaces, where the weight is a power of the distance to the boundary. Addeddate. 2013-09-21 04:02:58. The oblique derivative problem. Yu. V. Egorov, V. A. Kondrat'ev.Â Bengt Winzell, â€œA boundary value problem with an oblique derivativeâ€, Communications in Partial Differential Equations, 6:3 (1981), 305. Ulf Pillat, Bertâ€"Wolfgang Schulze, â€œSome classes of nonâ€"elliptic boundary value problems for pseudoâ€"differential operators II. overdetermined and underâ€" determined systemsâ€, Communications in Partial Differential Equations, 6:4 (1981), 373.Â Dian K. Palagachev, â€œNeutral PoincareÌ problem in Lp-Sobolev spaces: Regularity and Fredholmnessâ€, Internat Math Res Notices, 2006 (2006), 1. O. V. Besov, V. S. Vladimirov, V. V. Kozlov, S. M. Nikol'skii, Yu.